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Abstract

Preliminary to the study of the structure of an optical image from the standpoint of electromagnetic theory, an integral representation is obtained, for the electromagnetic field in the image space of an optical system. This representation, which is not restricted to systems of low angular aperture, is in the form of an angular spectrum of plane waves, and is closely related to that introduced by Luneberg (1944) as a vector generalization of well-known formulas of Debye (1909) and Picht (1925). It is shown that the representation has a simple physical interpretation in terms of a modified Huygens-Fresnel principle which operates with secondary plane waves rather then with secondary spherical waves.

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1. Introduction

Practically all investigations which relate to the structure of the optical image have, up to now, been almost exclusively based on geometrical optics or on the scalar diffraction theory. Valuable as these methods are in connection with the design of optical systems and the evaluation of their performance*, they suffer from two limitations, namely: (1) they do not give indications of the vectorial features of the image, i.e., the state of polarization of the light in the image region and the direction of the energy flow (Poynting vector); and (2) they cannot be easily applied to the problem of determining the structure of the image in systems of wide angular aperture, such as certain photographic lenses and some modern astronomical cameras. It would appear that more refined methods are needed for the treatment of such problems. Development of more powerful methods also seems desirable because of problems which arise in the related field of optics of microwaves. Microwave systems frequently employ numerical apertures which are considerably larger than those encountered in systems transmitting light, and it is not always clear to what extent the usual methods can be used with confidence [cf. Bachynski and Bekefi, 1956]. Moreover, in microwave optics the knowledge of the state of polarization of the image field is of direct interest.

In the present investigation an integral representation is obtained for the electromagnetic field in the image space of an optical system, which makes it possible to determine the complete structure of the image in systems of low as well as high angular aperture. Since the specification of the exact boundary conditions which the image field satisfies present formidable difficulties, the representation is an approximate one, but it may be expected to predict the field with a high degree of accuracy at all points which are not too close to the exit pupil. (The distance from the aperture must be large compared to the wavelength.) This is the case of the main interest in connection with practical applications. Our representation is intimately connected with that of Luneberg (1944) and may be regarded as a generalization of the well-known representation introduced for the scalar case by Debye (1909) and extended by Picht (1925). The present derivation appears to be

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*For review of the literature see Wolf (1951). A survey of recent research on the foundations of some of the methods is given in Wolf (1955).

simpler than Luneberg's and shows clearly the nature of the approximations that are involved when the solution is applied to problems of practical interest. It is also shown that the representation has a simple physical interpretation in terms of a modified Huygens-Fresnel principle which operates with (vectorial) secondary plane waves, rather than with secondary spherical waves.

In a subsequent paper the results will be used to determine the structure of electromagnetic images in the focal plane of aplanatic systems of low as well as high angular aperture.

2. An integral representation of the image field

Let S_o be a monochromatic point source of angular frequency ω, situated in the object space of a rotationally symmetrical system and let

$$(2.1) \qquad \vec{E}(P,t) = \mathcal{R}\,\vec{e}(P)e^{-i\omega t}, \quad \vec{H}(P,t) = \mathcal{R}\,\vec{h}(P)e^{-i\omega t}$$

denote the electric and magnetic vectors, at a point P in the image space, at time t. The vectors $\vec{e}$ and $\vec{h}$ which are to be determined are in general complex, and the symbol $\mathcal{R}$ denotes the real part.

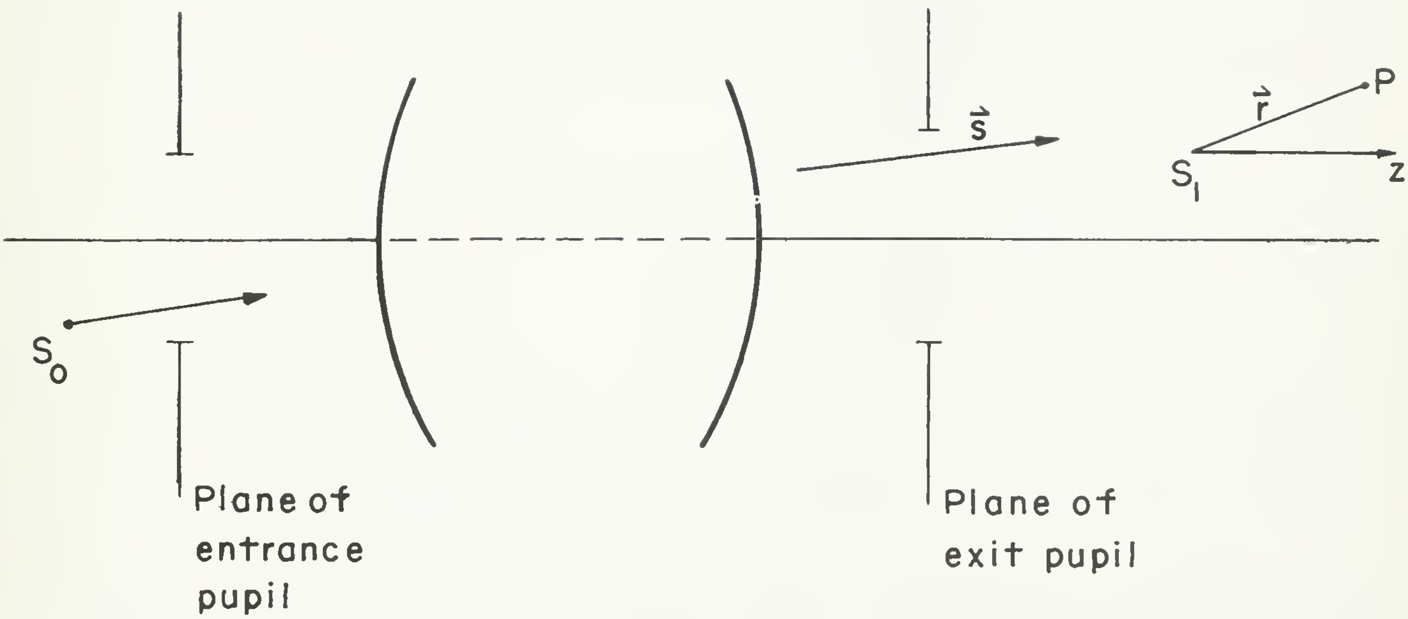
Let $\vec{s}$ denote a unit vector along a typical geometrical ray in the image field, i.e., a ray which reaches the image space through the exit pupil of the system. The positive direction of $\vec{s}$ is taken in the direction of the propagation of the light (Figure 1). We shall show that within an accuracy adequate for practical purposes, $\vec{e}(P)$ and $\vec{h}(P)$ for any point P in the image space which is not too close to the plane of the exit pupil may be expressed in the form

$$(2.2) \qquad \vec{e}(P) = \iint_{\Omega} \vec{A}(\vec{s})\, e^{ik\vec{s}\cdot\vec{r}} d\Omega, \quad \vec{h}(P) = \iint_{\Omega} \vec{B}(\vec{s}) e^{ik\vec{s}\cdot\vec{r}}\, d\Omega \quad .$$

Here $\vec{r}$ is the position vector of P, $\vec{A}(\vec{s})$ and $\vec{B}(\vec{s})$ are certain (generally complex) vector functions of $\vec{s}$ which are specified by the geometric optics field, and the integration is taken over the solid angle Ω formed by all the geometrical rays which reach the image space. The symbol k is the wave number in the image, i.e.,

$$(2.3) \qquad k = n\frac{\omega}{c} = \frac{2\pi}{\lambda} = n\frac{2\pi}{\lambda_o} = n\,k_o \;,$$

where n is the refractive index of the image space, c is the vacuum wavelength, λ and λ_o are the wavelengths in the image space and in the vacuum respectively, and k_o is the vacuum wave number.



<u>Figure 1.</u> Illustrating the notation

The integrals (2.2) represent the image field in the form of an angular spectrum of plane waves. A representation of this type is always possible provided that the unit vector $\vec{s}$ spans all directions including complex ones [cf. Bouwkamp (1954), 41-42]. These later directions represent surface waves or evanescent waves. For our purposes it will be sufficient to include only the real directions which are associated with rays in the image field of the instrument ($\vec{s}$ belonging to Ω).

Let x, y, z be the components of the position vector $\vec{r}$ (coordinates of P) with respect to Cartesian rectangular axes with origin at the Gaussian image point S_1 of S_o and with the positive z-direction at right angles to the plane of the exit pupil and pointing away from it (Figure 1). Further, let s_x, s_y, s_z be the components of $\vec{s}$. Since $\vec{s}$ is a unit vector, and the light is propagated into the image space ($s_z > 0$),

(2.4) $$s_z = \left|\sqrt{1 - s_x^2 - s_y^2}\right| .$$

The element $d\Omega$ of the solid angle is given by*

(2.5) $$d\Omega = \frac{ds_x ds_y}{s_z} ,$$

so that the first integral in (2.2) may be written as

(2.6) $$\vec{e}(P) = \iint_{\Omega} \vec{A}(s_x,s_y) \exp\left[ik(s_x x+s_y y+s_z z)\right] \frac{ds_x ds_y}{s_z} ;$$

there is, of course, a strictly similar expression for $\vec{h}(P)$.

*This follows, on transforming the usual expression

(I) $$d\Omega = \sin\theta \cos\theta \, d\phi$$

where (θ,ϕ) are the spherical polar angles. We have $s_x = \sin\theta\cos\phi$, $s_y = \sin\theta\sin\phi$, $s_z = \cos\theta$. Hence

(II) $$ds_x \, ds_y = \begin{vmatrix} \frac{\partial s_x}{\partial\theta} & \frac{\partial s_x}{\partial\phi} \\ \frac{\partial s_y}{\partial\theta} & \frac{\partial s_y}{\partial\phi} \end{vmatrix} d\theta d\phi$$

$$= \sin\theta\cos\theta \, d\theta d\phi = s_z \sin\theta \, d\theta d\phi ,$$

and (2.5) follows from (I) and (II).

To determine the vectors $\vec{A}$ and $\vec{B}$, consider first the behavior of the integral (2.6) when the distance

$$r = |\vec{r}| = \left|\sqrt{x^2 + y^2 + z^2}\right| \tag{2.7}$$

of the point of observation P from the origin of coordinates S_1 is very many wavelengths. Let $\vec{u}(u_x, u_y, u_z)$ be the unit vectors in the direction S_1P, i.e.,

$$u_x = \frac{x}{r}, \quad u_y = \frac{y}{r}, \quad u_z = \frac{z}{r} = \sqrt{1 - u_x^2 - u_y^2} \quad . \tag{2.8}$$

Using (2.7) and (2.8), the integral (2.6) may be written in the form

$$\vec{e}(P) = \iint_{\Omega} \frac{\vec{A}(s_x, s_y)}{s_z} \exp\left[ikr\, \phi(s_x, s_y;\, u_x, u_y)\right] ds_x ds_y \quad , \tag{2.9}$$

where

$$\begin{aligned} \phi(s_x, s_y;\, u_x, u_y) &= s_x u_x + s_y u_y + s_z u_z \\ &= s_x u_x + s_y u_y + \left|\sqrt{1 - s_x^2 - s_y^2}\right| \sqrt{1 - u_x^2 - u_y^2} \quad . \end{aligned} \tag{2.10}$$

Now when $kr = 2\pi r/\lambda >> 1$, the exponential term in (2.9) varies rapidly as (s_x, s_y) explores the domain of integration, and the real and imaginary parts will change sign many times. In consequence the contribution to the integral from the different elements of the domain of integration virtually cancel out; the only appreciable contribution comes from the elements (if any) centered on (s_x, s_y) in the domain of integration for which ϕ is stationary. This is the essence of the principle of stationary phase which may be used to find the asymptotic approximation to (2.9) for large values of kr. This approximation is derived in the Appendix at the end of this paper. It is shown there, that when $z < 0$, i.e., when P lies on the same side of the focal plane as the exit pupil, then

$$\begin{aligned} \vec{e}(P) &= -\frac{2\pi}{ik}\vec{A}(-u_x, -u_y)\,\frac{e^{-ikr}}{r} + \mathcal{O}\left(\frac{1}{kr}\right)^{3/2} && \text{when} \quad -\vec{u} \in \Omega \quad , \\ &= \mathcal{O}\left(\frac{1}{kr}\right)^{3/2} && \text{when} \quad -\vec{u} \notin \Omega \quad . \end{aligned} \tag{2.11}$$

The symbols $\in$ and $\notin$ stand for 'belongs to' and 'does not belong to' respectively, i.e., the first line in (2.11) applies when the direction PS_1 is parallel to one

of the geometrical rays which reaches the image space, and the second line applies when there is no such ray. When $z > 0$, i.e., when P lies on the side of the focal plane opposite to that of the exit pupil, one has, in place of (2.11),

$$(2.12)\qquad \vec{e}(P) = \frac{2\pi}{ik}\vec{A}(u_x,u_y)\frac{e^{ikr}}{r} + \mathcal{O}\left(\frac{1}{kr}\right)^{3/2} \qquad \text{when } \vec{u}\in\Omega ,$$

$$= \mathcal{O}\left(\frac{1}{kr}\right)^{3/2} \qquad \text{when } \vec{u}\notin\Omega .$$

Now the exact boundary conditions which the field vectors satisfy cannot be determined in practice. However, if the linear dimensions of the aperture are large compared to the wavelength, and if P is not too close to the aperture, a good approximation to $\vec{e}$ and $\vec{h}$ is obtained by imposing the approximate Kirchhoff's boundary conditions, i.e., the conditions

$$(2.13)\qquad \vec{e}(P') = \vec{e}^{(i)}(P'),\quad \vec{h}(P') = \vec{h}^{(i)}(P') \qquad \text{when } P'\in\mathcal{A} ,$$

$$\vec{e}(P') = 0 ,\quad \vec{h}(P') = 0 \qquad \text{when } P'\notin\mathcal{A} ,$$

where $\vec{e}^{(i)}(P')$, $\vec{h}^{(i)}(P')$ are the values of the unperturbed incident field at a point P' in the plane of the exit pupil $\mathcal{A}$. If we neglect terms of order $(kr)^{-3/2}$ we have from (2.11) and (2.13)

$$(2.14)\qquad \vec{A}(s_x,s_y) = -\frac{ik}{2\pi}\,\vec{e}^{(i)}(P')\,r'\,e^{ikr'} ,$$

where P' is the point in the direction $\vec{u} = -\vec{s}$ from S_1, and $r' \gg \lambda$ is the distance* S_1P'. Within the accuracy here in question P' may evidently be also identified with the point in the aperture through which the ray with direction of propagation $\vec{s} = -\vec{u}$ passes.

Except for points P' in the immediate neighborhood of the edge of the aperture, $\vec{e}^{(i)}(P')$ is, to a good approximation, given by [cf. Born and Wolf, 1958, Chapter III]

$$(2.15)\qquad \vec{e}^{(i)}(P') = \frac{\vec{a}}{\sqrt{R_1}\sqrt{R_2}}\exp\left[ik_0\mathcal{S}(P')\right] ,$$

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*In a typical case in light optics $r' \gtrsim 10$ cm, $\lambda \sim 5.10^{-5}$ cm so that $kr' = 2\pi r'/\lambda \gtrsim 10^6$.

where R_1 and R_2, are the principal radii of curvature of the geometrical wave front through P', $\mathcal{S}(P')$ is the eikonal function and $\vec{a}$ is a (generally complex) vector which is constant along the ray and perpendicular to it. We shall call $\vec{a}$ the <u>strength factor</u> of the ray (s_x, s_y). With a suitable choice of the phase constant of $\vec{a}$, $\mathcal{S}(P')$ represents the optical path from the source point S_o to P'. From (2.14) and (2.15)

$$\vec{A}(s_x,s_y) = -\frac{ik}{2\pi}\,\frac{r'}{\sqrt{R_1}\sqrt{R_2}}\,\exp\left[ik_o\left\{\mathcal{S}(P')+nr'\right\}\right] \quad . \tag{2.16}$$

We may express (2.14) in a somewhat different form which involves the wave aberration function of the system. Let σ be the Gaussian reference sphere, i.e., a sphere which has its center at the Gaussian image point S_1 of S_o and which passes through the center E of the exit pupil. Further, let W be the geometrical wave front through E. If F and G are the points in which $P'S_1$ intersects σ and W respectively, we have (see Figure 2),

$$\mathcal{S}(P') + nr' = [S_oG] + [GF] + [FS_1] \quad , \tag{2.17}$$

where the brackets $[....]$ denote optical lengths. Now since G and E lie on the same wave front, $[S_oG] = [S_oE]$. Also $[FS_1] = nR$ where R is the radius of the reference sphere. Hence

$$\mathcal{S}(P') + nr' = n(C + \Phi) \quad , \tag{2.18}$$

where

$$C = \frac{1}{n}[S_oE] + R = \text{constant} \tag{2.19}$$

and

$$\Phi = \Phi(s_x,s_y) = \frac{1}{n}[GF] \tag{2.20}$$

is the <u>aberration function</u>* of the system. Also, since the principal centers of curvature of the geometrical wave front lie in the region of the optical image, R_1, R_2 and r' can only differ by small relative amounts, so that the

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*The aberration is usually measured along the ray through F rather than along the radial direction SF_1. For practical purposes the two definitions may be taken as equivalent [cf. Wolf, 1952].

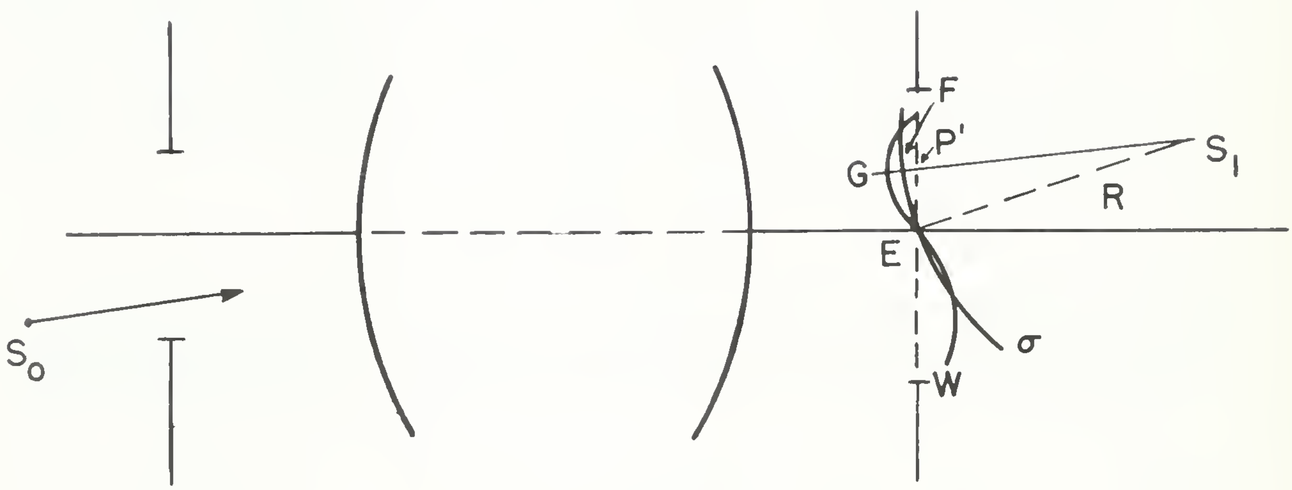

Figure 2. Definition of the aberration function.

term $r'/\sqrt{R_1}\sqrt{R_2}$ in (2.16) may be replaced by unity. If we also substitute from (2.18), (2.16) becomes

$$(2.21)\qquad \vec{A}(s_x,s_y) = -\frac{ik}{2\pi}\vec{a}(s_x,s_y)\exp\left[ik\left\{C+\Phi(s_x,s_y)\right\}\right] ,$$

and the diffraction integral (2.6) finally takes the form*

$$(2.22)\qquad \vec{e}(x,y,z) = -\frac{ik}{2\pi}e^{ikC}\iint_{\Omega}\frac{\vec{a}(s_x,s_y)}{s_z}\exp\left[ik\left\{\Phi(s_x,s_y)+s_x x+s_y y+s_z z\right\}\right]ds_x ds_y .$$

In a strictly similar manner the corresponding expression for $\vec{h}$ is obtained:

$$(2.23)\qquad \vec{h}(x,y,z) = -\frac{ik}{2\pi}e^{ikC}\iint_{\Omega}\frac{\vec{b}(s_x,s_y)}{s_z}\exp\left[ik\left\{\Phi(s_x,s_y)+s_x x+s_y y+s_z z\right\}\right]ds_x ds_y .$$

Here $\vec{b}$ is the magnetic strength factor, i.e., the vector which bears the same relation to the incident magnetic field $\vec{h}^{(i)}$ as $\vec{a}$ bears to $\vec{e}^{(i)}$ [cf. (2.15)]. It follows from Maxwell's equations that $\vec{a}$ and $\vec{b}$ obey the relation [cf. Born and Wolf (1958), § 3.1 eq. (12a) and (15a)]

$$(2.24)\qquad \vec{b} = \sqrt{\varepsilon/\mu}\ \vec{s}\wedge\vec{a} ,$$

where ε and μ denote the dielectric constant and the magnetic permeability of the image space, and the relation $n = \sqrt{\varepsilon\mu}$ is assumed. In our system of units (Gaussian) μ differs inappreciably from unity, but is retained here for the sake of symmetry.

The formulas (2.22) and (2.23) represent the solution of our problem. They express the image field as a superposition of plane waves of different directions of propagation. The 'complex amplitudes' $\vec{a}$, $\vec{b}$ of each constituent plane wave can be derived from a ray trace, taking into account the state of polarization of the field along each ray. The phase function Φ, which is a measure of the aberration of the system, may likewise be determined from the ray trace. It follows, in view of (2.24), that our solution satisfies the

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*Since the origin of the phase is arbitrary, we may, of course, set C = 0.

homogeneous Maxwell's equations

$$\text{curl } \vec{h} + ik_o \varepsilon \vec{e} = 0 \ , \tag{2.25}$$
$$\text{curl } \vec{e} - ik_o \mu \vec{h} = 0 \ ,$$

throughout the whole image space, and it is seen from (2.12) that it does obey the vectorial form of the Sommerfeld radiation condition [cf. C. Müller (1957), p. 13-14] in the half-space $z > 0$. Since the approximate Kirchhoff boundary conditions were used, our solution cannot be expected to represent the field adequately at points in the immediate neighborhood of the exit pupil. However, when the aperture is large compared to the wavelength, and when the point of observation $P(x,y,z)$ as well as the Gaussian focus S_1 are both situated at distances from the aperture that are also large compared to the wavelength, (2.22) and (2.23) may be expected to give a good approximation to the field.

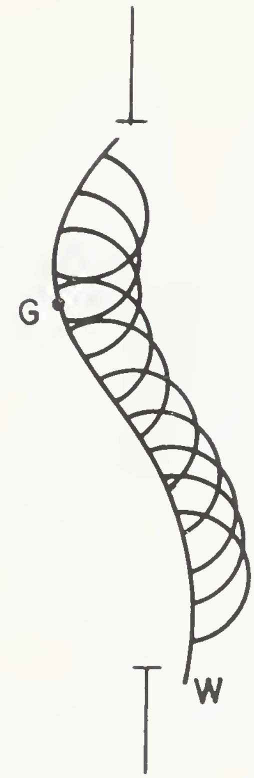
3. A physical interpretation of the diffraction integrals

The integrals (2.22) and (2.23) have a simple physical interpretation which is clearly brought out when comparing them with the corresponding integrals based on the application of the Huygens-Fresnel principle. According to the Huygens-Fresnel principle, each point G of the wave front W may be regarded as a center of a secondary disturbance which is propagated as a spherical wave of complex amplitude proportional to $\vec{e}^{(i)}(G)$, $\vec{h}^{(i)}(G)$. The total field at P is obtained by adding the effects of all the spherical waves which reach the image space through the exit pupil. Thus the Huygens-Fresnel approximation (suffix HF) for the field is

$$\vec{e}_{HF}(P) = \iint_W \vec{e}^{(i)}(G) \frac{e^{ikd}}{d} K(G) \, dW \ , \tag{3.1}$$
$$\vec{h}_{HF}(P) = \iint_W \vec{h}^{(i)}(G) \frac{e^{ikd}}{d} K(G) \, dW \ ,$$

where d is the distance GP and K is the usual inclination factor, which, for small obliquities, has the constant value $-ik/2\pi$. The integration extends over the part of the wave front which approximately fills the exit pupil (Figure 3(a)).

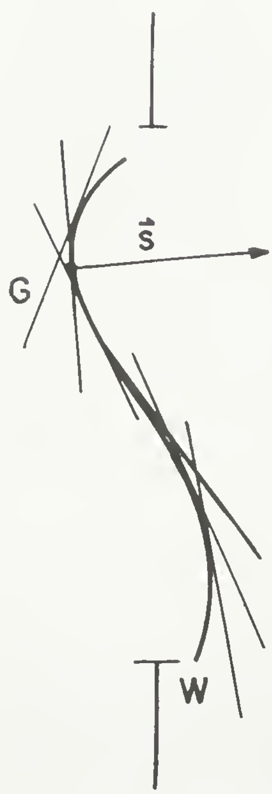
In the present formulation, the field is given by



(a) <u>Huygens' construction relating to the integrals (3.1).</u>

Spherical Secondary Waves

$$\vec{e}^{(i)}(G)\,\frac{e^{ikd}}{d}\,, \quad \vec{h}^{(i)}(G)\,\frac{e^{ikd}}{d}\,.$$



(b) <u>A construction relating to the integrals (3.2).</u>

Plane Secondary Waves

$$\vec{A}(\vec{s})e^{ik\vec{s}\cdot\vec{r}}, \quad \vec{B}(\vec{s})e^{ik\vec{s}\cdot\vec{r}}.$$

<u>Figure 3.</u> Physical interpretation of the diffraction integrals.

$$(3.2) \qquad \vec{e}(P) = \iint_{\Omega} \vec{A}(\vec{s})\, e^{ik\vec{s}\cdot\vec{r}}\, d\Omega \quad ,$$

$$\vec{h}(P) = \iint_{\Omega} \vec{B}(\vec{s})\, e^{ik\vec{s}\cdot\vec{r}}\, d\Omega \quad ,$$

i.e., as superposition of plane waves of complex amplitudes $\vec{A}(\vec{s})$, $\vec{B}(\vec{s})$, propagated in the $\vec{s}$-directions coinciding with the directions of all the geometrical rays that reach the image space. To each plane wave there belongs a plane wave front which is tangential to the wave front W at the appropriate point G (see Figure 3(b)). Thus each point of that portion of the wave front W which approximately fills the exit pupil may be regarded as giving rise to a secondary plane wave, and the field at P is obtained by adding the effects of all these secondary waves. If terms of order $(1/kr)^{3/2}$ are neglected, the vector amplitudes in the two constructions are, according to (2.11), related by

$$(3.3) \qquad \vec{e}^{(i)}(G) = -\frac{2\pi}{ik}\, \vec{A}(\vec{s})\, \frac{e^{-ikr}}{r} \quad ,$$

$$\vec{h}^{(i)}(G) = -\frac{2\pi}{ik}\, \vec{B}(\vec{s})\, \frac{e^{-ikr}}{r} \quad ,$$

where r is the distance S_1G.

Our diffraction integrals (2.22) and (2.23) are closely related to the diffraction integrals of Luneberg (1944) and may be regarded as generalizations of the integral introduced by Debye (1909) to represent a diffracted spherical scalar wave. A generalization of Debye's representation to non-spherical (scalar) waves was carried out by Picht (1925); this case was also treated recently by Focke (1956), who employed essentially the same method as used in the present paper.

Luneberg's integrals may be derived from our solution as follows: We have from (2.19) and (2.20),

$$(3.4) \qquad k\left\{C + \Phi\right\} = k_o\left\{\left[S_oE\right] + \left[GF\right] + \left[FS_1\right]\right\} \quad .$$

Let N be the foot of the perpendicular from S_1 onto the ray through G (Figure 4). In the asymptotic approximation, $[GF] + [FS_1]$ may evidently be replaced by the optical length $[GN]$ measured along the ray. Further, since G and E lie on the same wave front, $[S_oE] = [S_oG]$, and we may write in place of (3.4)

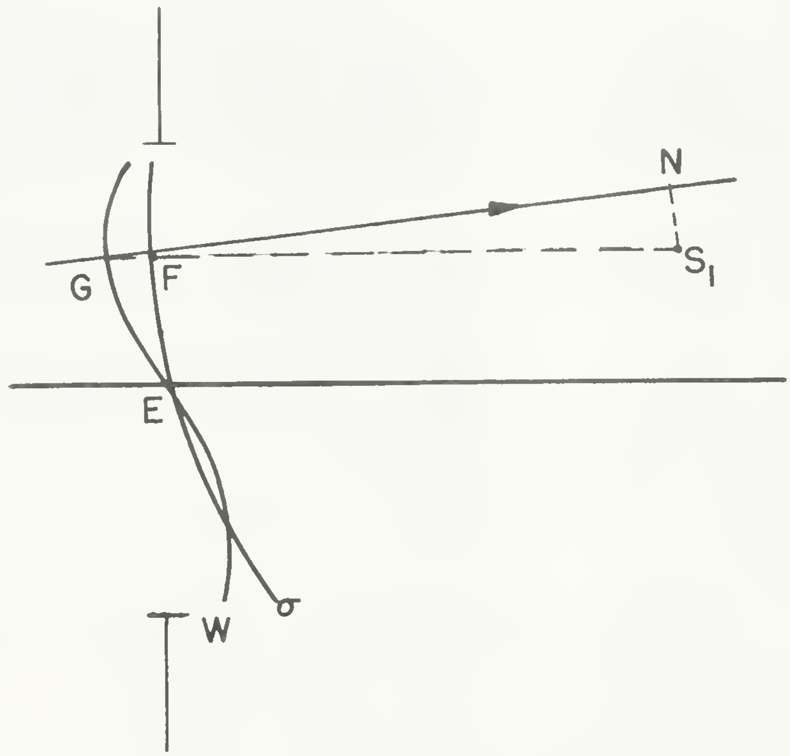


Figure 4: Transition to Luneberg's diffraction integrals.

$$(3.5) \qquad k\left\{C + \Phi\right\} = k_o\left\{\left[S_oG\right] + \left[GN\right]\right\} = k_o\left[S_oN\right] .$$

Let us now introduce in place of the unit vector $\vec{s}$ the 'ray vector'

$$(3.6) \qquad \vec{g} = n\,\vec{s} ,$$

and let p, q and m be its components, i.e.,

$$(3.7) \qquad p = ns_x, \quad q = ns_y, \quad m = ns_z = \left|\sqrt{n^2 - p^2 - q^2}\right| .$$

Then $\left[S_oN\right]$ becomes a function of p and q, and of the position of the source point S_o; it is the mixed characteristic function $W(S_o;\, p,q)$ of Hamilton. The strength factors $\vec{a}$ and $\vec{b}$, regarded as functions of p and q rather than of s_x

and s_y, will be denoted by $\overset{\approx}{a}$ and $\overset{\approx}{b}$ respectively, and the corresponding domain of integration, formed by the (p,q)-components of all the geometrical rays which reach the image space, will be denoted by $\tilde{\Omega}$. The integrals (2.22) and (2.23) become

$$(3.8) \qquad \vec{e}(x,y,z) = -\frac{ik_o}{2\pi} \iint_{\tilde{\Omega}} \frac{\overset{\approx}{a}(p,q)}{m} \exp\left[ik_o\left\{W(S_o;p,q) + px+qy+mz\right\}\right]dpdq ,$$

$$(3.9) \qquad \vec{h}(x,y,z) = -\frac{ik_o}{2\pi} \iint_{\tilde{\Omega}} \frac{\overset{\approx}{b}(p,q)}{m} \exp\left[ik_o\left\{W(S_o;p,q) + px+qy+mz\right\}\right]dpdq .$$

In place of (2.24) we now have

$$(3.10) \qquad \overset{\approx}{b} = \frac{1}{\mu} \vec{g} \wedge \overset{\approx}{a} .$$

The formulas (3.8) and (3.9) are formally identical with Luneberg's diffraction integrals. They were obtained by Luneberg not as an approximate solution of the true physical problem, but rather as an exact solution to an idealized problem, namely the following: to find a solution of the homogeneous Maxwell's equations, valid throughout the whole space, which has a prescribed behavior at infinity. In spite of the mathematical elegance of Luneberg's formulation, we prefer to regard our integrals as an approximate solution of the true physical problem, valid at distances from the aperture that are large compared to the wavelength. Our analysis shows that the integrals (2.22) and (2.23) [or (3.8) and (3.9)] are then effectively equivalent to the solution obtained by a direct application of the (vectorial) Huygens-Fresnel-Kirchhoff diffraction theory. However, since our formulas do not involve an inclination factor, they are more suitable for the treatments of problems involving large obliquities, e.g., the problem of determining the structure of an optical image in a system with a high angular aperture. This problem will be discussed in another paper.

Appendix

Asymptotic evaluation of the integral (2.9)

We derive here the asymptotic approximation for large kr to the integral (2.9), viz.,

$$\text{(A1)} \qquad \vec{e} = \iint_{\Omega} \frac{\vec{A}(s_x,s_y)}{s_z} \exp\left[ikr\phi(s_x,s_y;\, u_x,u_y)\right] ds_x ds_y \quad ,$$

where

$$\text{(A2)} \qquad \phi(s_x,s_y;\, u_x,u_y) = s_x u_x + s_y u_y + s_z u_z \quad ,$$

$$\text{(A3)} \qquad s_z = \left|\sqrt{1 - s_x^2 - s_y^2}\right| \quad ,$$

$$\text{(A4)} \qquad u_z = \sqrt{1 - u_x^2 - u_y^2} \quad .$$

As explained in Section 2, the asymptotic approximation may be obtained by the application of the principle of stationary phase. The principle is well known in connection with single integrals; its generalization to double integrals of a wide class of practical interest was carried out by Focke (1954), who determined the whole asymptotic expansion of these integrals. Other ways of deriving the asymptotic expansions were discussed by Jones and Kline (1956), and Chako (1958). It follows from these investigations, that for large kr, (A1) reduces to*

$$\text{(A5)} \qquad \vec{e} = \frac{2\pi i}{kr} \sum \frac{j}{\sqrt{|\Delta|}} \frac{\vec{A}(s_x',s_y')}{s_z'} \exp\left[ikr\phi(s_x',s_y';u_x,u_y)\right] + \mathcal{O}\left(\frac{1}{kr}\right)^{3/2} .$$

Here s_x', s_y' are the values of s_x and s_y for which ϕ is stationary within the domain of integration, i.e., they are the roots of the equation

$$\text{(A6)} \qquad \frac{\partial\phi}{\partial s_x} = \frac{\partial\phi}{\partial s_y} = 0 \quad ;$$

---

*Strictly speaking, the 'amplitude function' in the integrals considered by these authors is a real scalar, not a complex vector as in (A1), but the generalization to the present case is quite trivial.

further,

(A7) $$\Delta = \left[\frac{\partial^2\phi}{\partial s_x^2}\frac{\partial^2\phi}{\partial s_y^2} - \left(\frac{\partial^2\phi}{\partial s_x \partial s_y}\right)^2\right]' ,$$

(A8) $$j = \begin{cases} 1 & \text{when } \Delta > 0 , \quad \left(\frac{\partial^2\phi}{\partial s_x^2}\right)' > 0 , \\ -1 & \text{when } \Delta > 0 , \quad \left(\frac{\partial^2\phi}{\partial s_x^2}\right)' < 0 , \\ i & \text{when } \Delta < 0 , \end{cases}$$

and the prime in (A7) and (A8) denotes values at the stationary point. The summation in (A5) is taken over all the stationary points.

If there is no stationary point within the domain Ω of integration, then

(A9) $$\vec{e} = \mathcal{O}\left(\frac{1}{kr}\right)^{3/2} .$$

Now we have from (A2) and (A3),

(A10) $$\frac{\partial\phi}{\partial s_x} = u_x + u_z \frac{\partial s_z}{\partial s_x} = u_x - u_z \frac{s_x}{s_z} ,$$

with a similar expression for $\partial\phi/\partial s_y$. It follows, if (A4) is also used, that ϕ is stationary when

(A11) $$s_x' = u_x, \quad s_y' = u_y, \quad s_z' = u_z \qquad \text{if } u_z > 0, \quad \text{(a)}$$

$$s_x' = u_x, \quad s_y' = u_y, \quad s_z' = u_z \qquad \text{if } u_z < 0. \quad \text{(b)}$$

Thus, if $u_z > 0$, there is one stationary point which contributes to the asymptotic approximation of $\vec{e}$, provided that $\vec{u}$ lies within Ω; if $u_z < 0$, the same is true, provided that $-\vec{u}$ lies in Ω. If u_z (or $-u_z$, as the case may be) does not lie in Ω, ϕ is not stationary within the domain of integration, and, by (A9) $\vec{e}$ is then of the order of $(kr)^{-3/2}$.

From (A10) and the corresponding relation for $\partial\phi/\partial s_y$, we have, on using (A11),

(A12) $$\left(\frac{\partial^2\phi}{\partial s_x^2}\right)' = \mp\left(1+\frac{u_x^2}{u_z^2}\right), \quad \left(\frac{\partial^2\phi}{\partial s_y^2}\right)' = \mp\left(1+\frac{u_y^2}{u_z}\right), \quad \left(\frac{\partial^2\phi}{\partial s_x \partial s_y}\right)' = \mp\frac{u_x u_y}{u_z^2},$$

where the upper sign is taken when $u_z > 0$ and the lower sign, when $u_z < 0$. The expressions (A7) and (A8) become

(A13) $$\triangle = \frac{1}{u_z^2},$$

(A14) $$j = \mp 1 \qquad \text{according as } u_z \gtrless 0 .$$

We also have

(A15) $$\vec{A}(s_x', s_y') = \vec{A}(\pm u_x, \pm u_y),$$

(A16) $$\phi(s_x', s_y'; u_x, u_y) = \pm 1,$$

the upper or lower sign again being taken according as $u_z \gtrless 0$.

From (A5) and (A9) we thus finally obtain the following asymptotic approximations to $\vec{e}$ for large kr:

(A17) $$\left.\begin{aligned} \vec{e} &= \frac{2\pi}{ik}\vec{A}(u_x,u_y)\frac{e^{ikr}}{r} + \mathcal{O}\left(\frac{1}{kr}\right)^{3/2} && \text{when } u_z > 0 \text{ and } \vec{u}\in\Omega, \\ &= \mathcal{O}\left(\frac{1}{kr}\right)^{3/2} && \text{when } u_z > 0 \text{ and } \vec{u}\notin\Omega, \end{aligned}\right\} \text{(a)}$$

$$\left.\begin{aligned} \vec{e} &= -\frac{2\pi}{ik}\vec{A}(-u_x,-u_y)\frac{e^{-ikr}}{r} + \mathcal{O}\left(\frac{1}{kr}\right)^{3/2} && \text{when } u_z < 0 \text{ and } -\vec{u}\in\Omega, \\ &= \mathcal{O}\left(\frac{1}{kr}\right)^{3/2} && \text{when } u_z < 0 \text{ and } -\vec{u}\notin\Omega. \end{aligned}\right\} \text{(b)}$$

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